

# COMPUTING KAZHDAN CONSTANTS BY SEMIDEFINITE PROGRAMMING

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**ABSTRACT.** Kazhdan constants of discrete groups are hard to compute and the actual constants are known only for several classes of groups.

By solving a semidefinite programming problem by a computer, we obtain a lower bound of the Kazhdan constant of a discrete group. Positive lower bounds imply that the group has property (T).

We study lattices on  $\tilde{A}_2$ -buildings in detail. For  $\tilde{A}_2$ -groups, our numerical bounds look identical to the known actual constants. That suggests that our approach is effective.

For a family of groups,  $G_1, \dots, G_4$ , that are studied by Ronan, Tits and others, we conjecture the spectral gap of the Laplacian is  $(\sqrt{2} - 1)^2$  based on our experimental results.

For  $\mathrm{SL}(3, \mathbb{Z})$  and  $\mathrm{SL}(4, \mathbb{Z})$  we obtain lower bounds of the Kazhdan constants, 0.2155 and 0.3285, respectively, which are better than any other known bounds. We also obtain 0.1710 as a lower bound of the Kazhdan constant of the Steinberg group  $\mathrm{St}_3(\mathbb{Z})$ .

## 1. INTRODUCTION

Property (T) of a group was introduced by Kazhdan and he proved that, for example,  $\mathrm{SL}(n, \mathbb{Z})$  have property (T) if  $n \geq 3$ . We say a group is a Kazhdan group if the group has property (T). Property (T) plays an important role in many areas of mathematics.

For a Kazhdan group, given a finite generating set, a positive constant  $\kappa$ , called *the Kazhdan constant* is defined. Lower and upper bounds of the Kazhdan constants are known for some examples of groups, but to compute the value of the Kazhdan constant is usually hard.

Recently Ozawa [Oz] found a new characterization of a Kazhdan group among finitely generated groups. A significance of his theorem is that it gives a semidefinite algorithm to decide if a finitely generated group is a Kazhdan group or not. To be precise, he gives a description of the spectral gap of the Laplacian, which is closely related to the Kazhdan constant.

Netzer-Thom implemented it as a computer program using Semidefinite programming. They obtain a lower bound, 0.1783, for the Kazhdan constant of  $\mathrm{SL}(3, \mathbb{Z})$  with respect to the set of elementary matrices, which is much better than any other known bounds.

Building up on those ideas, we compute lower bounds of the spectral gaps of the Laplacian and the Kazhdan constants in various examples. As far

as we know, among infinite groups, the Kazhdan constants are known only for the family of “ $\tilde{A}_2$ -groups” (Theorem 3.2). Those are groups that act on  $\tilde{A}_2$ -buildings properly and cocompactly in a very special way. For those groups, we obtain lower bounds of the Kazhdan constants, which are almost identical to the actual values (Tables 1 and 2).

We regard this computation as an evidence that our method is effective to compute a good lower bound.

We also point out that this class of groups is interesting in various ways, for example, the equality holds in two basic inequalities regarding the spectral gaps and the Kazhdan constants (see Corollary 3.3).

We then discuss another class of groups,  $G_1, \dots, G_4$ , which also act on some  $\tilde{A}_2$ -buildings. This family is found by Ronan [R], then studied from various viewpoints, including by Tits [Ti] as “triangles of groups” (see Section 3.4). In particular they are Kazhdan groups, but the Kazhdan constants are unknown. We obtain 0.239146... as a common (numerical) lower bound of the Kazhdan constants (see Table 3), and also 0.171573 as a common (numerical) lower bound of the spectral gaps. We predict that they are  $(\sqrt{2} - 1)/\sqrt{3} = 0.239146...$  and  $(\sqrt{2} - 1)^2 = 0.1715728...$ .

Also, we obtain 0.2155 as a lower bound of the Kazhdan constant of  $\mathrm{SL}(3, \mathbb{Z})$ , which is slightly better(bigger) than the one by Netzer-Thom, and also 0.3285 for  $\mathrm{SL}(4, \mathbb{Z})$ , which is much better than any other known bounds (see Table 4).

We were not able to obtain a positive bound for  $\mathrm{SL}(5, \mathbb{Z})$  because of the lack of the power of the computer. Also, unfortunately, we did not find any new examples of Kazhdan groups by our method.

The Steinberg group  $\mathrm{St}_n(\mathbb{Z})$  is very closely related to  $\mathrm{SL}(n, \mathbb{Z})$ , which is the quotient of  $\mathrm{St}_n(\mathbb{Z})$  by  $\mathbb{Z}/2\mathbb{Z}$ . We obtain 0.1710 as a lower bound of the Kazhdan constant of  $\mathrm{St}_3(\mathbb{Z})$  (see the inequality 4.1). As far as we know, no concrete lower bounds of the Kazhdan constants of the Steinberg groups were known. Also, our bound is obtained from a finite presentation of  $\mathrm{St}_3(\mathbb{Z})$ . In that sense, this is a new proof that  $\mathrm{St}_3(\mathbb{Z})$  is a Kazhdan group.

Then we turn our attention to finite groups. All of them are Kazhdan groups, but again, there are a small number of cases that the Kazhdan constants are known. One such example is the family of finite Coxeter groups. We obtain bounds and they are very close to the known values (see Section 5.1). We also obtain lower bounds for complex Coxeter groups, for which the actual values are not known (see Section 5.2). We believe that our bounds are very close to the actual values.

At the end, we ask questions that arise naturally from our experimental results.

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## 2. KAZHDAN CONSTANTS AND SPECTRAL GAPS

Let  $\Gamma$  be a finitely generated group with a finite generating set  $S$ . For a Hilbert space  $\mathcal{H}$  and a unitary representation  $\pi : \Gamma \rightarrow \mathcal{H}$ , set

$$\kappa(\Gamma, S, \pi) = \inf_{\substack{\xi \in \mathcal{H}, \\ \|\xi\|=1}} \max_{s \in S} \|\pi(s)\xi - \xi\|.$$

We define the (*optimal*) *Kazhdan constant* with respect to  $S$  by

$$\kappa(\Gamma, S) = \inf \{ \kappa(\Gamma, S, \pi) \mid \pi \text{ has no non-zero invariant vector} \}.$$

If  $\kappa(\Gamma, S) > 0$  for some (then any) finite generating set  $S$ , then the group  $\Gamma$  is called a *Kazhdan group*.

From now on, we assume that a generating set  $S$  is symmetric. Let  $\mathbb{R}[\Gamma]$  be the group ring of  $\Gamma$  with coefficients in  $\mathbb{R}$ . Define the *unnormalized Laplacian*  $\Delta$  by

$$\Delta = |S| - \sum_{s \in S} s = \frac{1}{2} \sum_{s \in S} (1 - s)(1 - s^{-1}) \in \mathbb{R}[\Gamma].$$

Define the  $*$ -operation on  $\mathbb{R}[\Gamma]$  by  $(\sum_{g \in \Gamma} r_g g)^* = \sum_{g \in \Gamma} r_g g^{-1}$  ( $r_g \in \mathbb{R}$ ). By definition,  $\Delta^* = \Delta$ . For any unitary representation  $\pi : \Gamma \rightarrow U(\mathcal{H})$ ,  $\pi(\Delta)$  is an operator on  $\mathcal{H}$ . We remark that the normalized Laplacian is defined as  $\Delta/|S|$ , which is used in [Oz].

**Theorem 2.1** (Proposition 5.4.5 and Remark 5.4.7 of [BdlHV]). *Let  $\Gamma$  be a discrete group with finite symmetric generating set  $S$ . Suppose there exists  $\varepsilon > 0$  such that for any unitary representation  $\pi : \Gamma \rightarrow \mathcal{H}$  with no non-zero invariant vector,*

$$\langle \pi(\Delta)\xi, \xi \rangle \geq \varepsilon \langle \xi, \xi \rangle$$

*for any  $\xi \in \mathcal{H}$ . Then*

$$(2.1) \quad \sqrt{\frac{2\varepsilon}{|S|}} \leq \kappa(\Gamma, S).$$

*In particular,  $\Gamma$  is a Kazhdan group.*

In some cases, the equality holds in the inequality 2.1 (see Corollary 3.3).

We remark [BdlHV] uses the normalized Laplacian, therefore the factor  $1/|S|$  does not appear in their setting. We call  $\varepsilon$  a *spectral gap* and the supremum of  $\varepsilon$  the *spectral gap* of  $\Delta$ .

**2.1. Ozawa's criterion.** We recall a theorem by Ozawa.

**Theorem 2.2** (Ozawa [Oz, Main Theorem]). *There exist  $b_1, \dots, b_n \in \mathbb{R}[\Gamma]$  and  $\varepsilon > 0$  such that*

$$(2.2) \quad \Delta^2 - \varepsilon \Delta = \sum_{i=1}^n b_i^* b_i,$$

*if and only if  $\Gamma$  is a Kazhdan group.*

Moreover, if  $\Gamma$  is a Kazhdan group then for any rational number  $\varepsilon' \leq \varepsilon$ , there exist positive rational numbers  $r_i$  and  $b_1, \dots, b_n \in \mathbb{Q}[\Gamma]$  such that  $\Delta^2 - \varepsilon' \Delta = \sum_{i=1}^n r_i b_i^* b_i$ .

By the spectral mapping theorem, the condition in Theorem 2.1 and the condition in Theorem 2.2 are equivalent for a same constant  $\varepsilon$  ([Oz]).

**Definition 2.3** (Spectral gap). We call  $\varepsilon$  and the supremum of  $\varepsilon$ , denoted by  $\varepsilon(\Gamma, S)$ , in Theorem 2.2 a *spectral gap* and the *spectral gap* (in the sense of Ozawa), respectively. We sometimes write  $\varepsilon(\Gamma, S)$  as  $\varepsilon$ .

Notice that from the moreover part of the theorem, since  $\Gamma$  is countable, if  $\Gamma$  is a Kazhdan group, we will be able to find a rational solution with  $\varepsilon > 0$ , which implies that  $\Gamma$  is a Kazhdan group. This is the case even if a group does not have a solvable word problem since we only claim the algorithm is semidefinite (ie, stops only when there is a positive solution).

Also a number  $\varepsilon$  we obtain gives a lower bound of the spectral gap, which will give a lower bound of the Kazhdan constant by the inequality (2.1).

**2.2. Solving the equation.** In view of this, we try to find a (rational) solution,  $\varepsilon, b_i$ , for the equation (2.2) using a computer to obtain a lower bound of the Kazhdan constant. Netzer-Thom [NT] already carried out this strategy and found a solution with positive  $\varepsilon$  for  $\mathrm{SL}(3, \mathbb{Z})$ . The lower bound they obtain for the Kazhdan constant is much better than any known bounds (see Section 4)

There is an issue in this strategy. A computer can find only a numerical solution, so that even if we find a solution with some  $\varepsilon > 0$ , maybe the equation does not have any “exact” solutions with  $\varepsilon > 0$ . Netzer-Thom addressed this issue and found a way to certify the existence of an exact solution, with an estimate of  $\varepsilon$ , once we have a “good” numerical solution as the following lemmas show.

**Lemma 2.4** (Netzer-Thom [NT]). *Let  $\Gamma$  be a group with a finite generating set  $S = S^{-1}$ . Let  $c = \sum_g c_g g$  be an element of  $\mathbb{R}[\Gamma]$  satisfying  $\sum_g c_g = 0$  (i.e.  $c$  is in the augmentation ideal) and  $c^* = c$ . Let  $D > 0$  be an integer such that if  $c_g \neq 0$ , then  $g$  is a product of at most  $2^D$  elements from  $S$ . Then*

$$c + 2^{2D-1} \|c\|_1 \cdot \Delta \in \left\{ \sum_{i=1}^n b_i^* b_i \mid b_1, \dots, b_n \in \mathbb{R}[\Gamma] \right\},$$

where  $\|c\|_1 = \sum_g |c_g|$ . Moreover, if  $S$  does not contain self-inverse elements, then even

$$c + 2^{2D-2} \|c\|_1 \cdot \Delta \in \left\{ \sum_{i=1}^n b_i^* b_i \mid b_1, \dots, b_n \in \mathbb{R}[\Gamma] \right\}.$$

Using this lemma we explain how we possibly obtain an exact solution from a numerical solution. The following lemma is implicit in [NT].

**Lemma 2.5.** *Suppose a constant  $\varepsilon > 0$  and  $b_1, \dots, b_m \in \mathbb{R}[\Gamma]$  are given. Assume  $\sum_{i=1}^m b_i^* b_i$  is in the augmentation ideal. Set*

$$c = \Delta^2 - \varepsilon \Delta - \sum_{i=1}^m b_i^* b_i.$$

*Suppose  $c = \sum_g c_g g$ . Let  $D > 0$  be an integer such that if  $c_g \neq 0$ , then  $g$  is a product of at most  $2^D$  elements from  $S$ . Assume  $\varepsilon - 2^{2D-1} \|c\|_1 > 0$ . Then,  $\Gamma$  is a Kazhdan group and  $\varepsilon - 2^{2D-1} \|c\|_1$  is a spectral gap.*

*Proof.* Notice that  $c$  is in the augmentation ideal. Since  $c^* = c$ , we can apply Lemma 2.4 to  $c$ , and there must exist  $b_{m+1}, \dots, b_{m+m'}$  such that  $c + 2^{2D-1} \|c\|_1 \cdot \Delta = \sum_{i=m+1}^{m+m'} b_i^* b_i$ . Plug in the definition of  $c$  to this, and get

$$\Delta^2 - (\varepsilon - 2^{2D-1} \|c\|_1) \Delta = \sum_{i=1}^{m+m'} b_i^* b_i.$$

This implies the conclusion.  $\square$

Now once we obtain a numerical solution  $\varepsilon, b_i$ , then we apply Lemma 2.5 to the solution after we modify  $b_i$  (usually, slightly) so that  $c$  is in the augmentation ideal. Notice that we have a more chance to have  $\varepsilon - 2^{2D-1} \|c\|_1 > 0$  if  $\varepsilon$  is larger and  $D$  and  $\|c\|_1$  are small. The moreover part of Lemma 2.4 could be used to improve the estimate, but it will not be so critical in our experiments.

**2.3. Semidefinite programming.** Although we can apply Lemma 2.5 to any tuples of  $\varepsilon, b_1, \dots, b_m$ , we have a better chance to succeed if we start with a good numerical solution. We briefly explain how we find a solution by a computer. Following [NT] we use *Semidefinite programming* (SDP) to find a solution. We refer interested readers to their paper for details.

Here we explain only the point that is important for us. To set up and solve an optimization problem by SDP, we first fix a positive integer  $d$  that is an upper bound of the *support range* of solutions  $b_i$ 's of the equation 2.2, namely, the word length of the group elements in  $b_i$ 's that have non-zero coefficients is at most  $d$ . For each  $d > 0$ , we solve an optimization problem to maximize  $\varepsilon \geq 0$  such that a solution in the support range  $\leq d$  exists for the equation 2.2.

There are three reasons why the algorithm for a given  $d$  does not stop: (i) a group is not a Kazhdan group; (ii) it is a Kazhdan group but a solution with the support range  $\leq d$  does not exist with  $\varepsilon > 0$ ; or (iii) a solution exists in that support range but the computer does not have enough power to find a solution. The bigger the support range  $d$  is, the more chance there is that the computation will not be finished. Also, even if the algorithm stops for some  $d$  and gives a positive  $\varepsilon$ , maybe there is a solution for bigger  $d$  that gives a larger  $\varepsilon$ , so that our  $\varepsilon$  is smaller than the spectral gap. See Section 6 for an example of the numerical result that shows different bounds

for different  $d$ . Interestingly, it turns out that sometimes, the  $\varepsilon$  we obtain for a small  $d$  is very close to the actual spectral gap (for example, Table 1).

We have a heuristic estimate on  $d$ . If the longest relators in a given presentation of a group have the word length  $< 4\ell$ , then our experiments tend to work if we take  $d = \ell$ . (Notice that the word length of the group elements that appear in the equation 2.2 is at most  $2d$ . We then identify two such words as group elements using a relator of length at most  $2d + 2d = 4d$ . So,  $d$  should be at least as big as  $\ell$ , otherwise there are relators that are not taken into account.) For example, in Section 3.2, the relators for the group  $G_T$  has length 3, and  $d = 1$  works for the experiments.

Compared to [NT] there is one extra ingredient in our approach. To obtain a lower bound of the Kazhdan constant of  $\mathrm{SL}(3, \mathbb{Z})$ , only integer matrices and rational numbers appear in the algorithm to solve the equation 2.2. So, a computation is done without dealing with a presentation of the group. But we start with a finite presentation of a group (except for  $\mathrm{SL}(n, \mathbb{Z})$  and finite groups) then try to solve the equation 2.2. For that, as a part of the algorithm, we (have to) replace a product of generators with another product in the right hand side of the equation properly using a relation of the group. But we do not use/need the solution of the word problem of the group. Notice that once we find a solution of the equation 2.2, even if without completely solving the word problem on the way, the  $\varepsilon$  we get is a lower bound for the group. Possibly this part of the algorithm (ie, how much we do the replacement) may affect its efficiency. But mathematically speaking, the idea is elementary and we skip details.

**2.4. About tables.** The rest of the paper is the result of our computer experiments on various examples of groups. We make tables to present our results. It is mostly about lower bounds of the Kazhdan constants,  $\kappa$ .

We first numerically find the maximal  $\varepsilon$  for which the equation (2.2) has a solution  $\{b_i\}$ . If  $\varepsilon > 0$ , then we plug it in to the left hand side of the inequality (2.1). In this way we obtain the “numerical (lower) bound” of  $\kappa$ .

If the support range of  $\{b_i\}$  is  $d$ , then we choose  $D$  such that  $2d \leq 2^D$ . Then we can apply Lemma 2.5 to the  $\varepsilon > 0$  and the solutions  $\{b_i\}$ , and obtain a “certified” lower bound,  $\varepsilon - 2^{2D-1} \|c\|_1$ , of the spectral gap. If this is positive, it will give a “certified (lower) bound” of  $\kappa$  using the inequality (2.1).

We occasionally mention known lower/upper bounds of  $\kappa$ , also the exact values of  $\varepsilon, \sqrt{2\varepsilon/|S|}, \kappa$  to compare with our bounds. Also we sometimes mention the support range  $d$  for which we find the numerical solution  $\varepsilon, b_i$ , as well as  $m$  that is the number of the elements  $b_i$ ’s.

### 3. LATTICES ON $\tilde{A}_2$ -BUILDINGS

The first examples of our experiment are *uniform lattices* on  $\tilde{A}_2$ -buildings, namely, finitely generated groups that act on  $\tilde{A}_2$ -buildings by automorphisms, properly and cocompactly. A general reference is [BdlHV, S5.7].

It is known that those groups are Kazhdan groups. Moreover, the Kazhdan constants are known for a certain family of groups with certain generating sets. As far as we know this is the only case where the Kazhdan constants are known for infinite groups. We will compare them with our lower bounds.

**3.1.  $\tilde{A}_2$ -buildings.** Let  $\Pi$  be a finite projective plane of order  $q \geq 2$  with a set  $P$  of points, a set  $L$  of lines and an incidence relation between lines and points. Each point (line) is incident with  $q + 1$  lines (points, resp.). Also  $|P| = |L| = q^2 + q + 1$ .

The *incident graph* is a bipartite graph whose vertex set is  $P \cup L$  such that there is an edge between  $p \in P$  and  $\ell \in L$  if  $p, \ell$  are incident.

An  $\tilde{A}_2$ -*building* is a 2-dimensional, simply connected, connected simplicial complex such that the link of any vertex is the incidence graph of a finite projective plane.

The most familiar example of a projective plane is the projective plane  $PG(2, F)$  over a field  $F$  (those are called *Desarguesian plane*), namely, form a 3-dimensional vector space over  $F$ , letting  $P, L$  be the sets of 1- and 2-dimensional subspaces with incidence being inclusion. If  $|F| = q$ , then  $PG(2, F)$  is also denoted by  $PG(2, q)$ . (cf. [CMS]).

The easiest example is when  $F = F_2$ . In this case, the order of  $PG(2, 2)$  is 2, so that the incidence graph is a bipartite graph with 14 vertices and the degree of each vertex is 3. The graph is called the *Heawood graph*.

We quote a theorem. The first assertion is known in various forms (Pansu [Pan], Zuk [Zuk], Ballmann-Świątkowski [BS]). For the bound, see [BdlHV, Theorem 5.7.7]. A finite symmetric generating set  $S$  is explicitly given (see (Pvi) in [BdlHV, Section 5.4]).

**Proposition 3.1.** *Let  $X$  be an  $\tilde{A}_2$ -building and suppose a discrete group  $G$  acts on  $X$  by automorphisms, property and co-compactly. Then  $G$  is a Kazhdan group.*

*Moreover, if the links of the vertices of  $X$  are the incidence graphs of the finite projective planes of order  $q$ , then there is a finite generating set  $S$  of  $G$  such that  $\sqrt{\frac{2(\sqrt{q}-1)^2}{(\sqrt{q}-1)^2+\sqrt{q}}}$  is a Kazhdan constant.*

In some cases,  $\sqrt{\frac{2(\sqrt{q}-1)^2}{(\sqrt{q}-1)^2+\sqrt{q}}}$  is the Kazhdan constant for  $S$  (see Corollary 3.3).

**3.2.  $\tilde{A}_2$ -groups and triangle presentation.** We review one way to construct uniform lattices of  $\tilde{A}_2$ -buildings following [CMS]. Let  $\Pi = (P, L)$  be a finite projective plane and  $\lambda : P \rightarrow L$  be a bijection. A *triangle presentation* compatible with  $\lambda$  is a set  $T$  of triples  $(x, y, z), x, y, z \in P$  such that

(A) given  $x, y \in P$ , then  $(x, y, z) \in T$  for some  $z \in P$  if and only if  $y, \lambda(x)$  are incident;

(B)  $(x, y, z) \in T$  implies  $(y, z, x) \in T$ ;

(C) given  $x, y \in P$ , then  $(x, y, z) \in T$  for at most one  $z \in P$ .

Given a triangle presentation  $T$ , we define a group  $G_T$ , called an  $\tilde{A}_2$ -group, as follows:

$$G_T = \langle \{a_x\}_{x \in P} \mid a_x a_y a_z = 1 \text{ if } (x, y, z) \in T \rangle.$$

The set of  $a_x$  and their inverses (which are labeled by  $L$  such that  $a_x^{-1}$  is by  $\lambda(x) \in L$ ) is called *the set of natural generators* (from the triangle presentation).

Then the Cayley graph of  $G_T$  is (the 1-skeleton of) a (“thick”)  $\tilde{A}_2$ -building such that the link of every vertex is the incidence graph of  $(P, L)$ . So,  $G_T$  acts on the building properly and cocompactly. Moreover, the set of natural generators will be  $S$  in Proposition 3.1.

In [CMSZ2], they found all triangle presentations (and  $\lambda$ ) in the case that  $(P, L)$  is the projective planes  $PG(2, q)$  for  $q = 2, 3$ .

In fact a converse holds, [CMSZ]: a group is an  $\tilde{A}_2$ -group if it acts freely and transitively on the vertices of an  $\tilde{A}_2$ -building, and if it induces a cyclic permutation of the “type” of the vertices (there are three types for vertices of a  $\tilde{A}_2$ -building).

[CMS] obtained the Kazhdan constant for a class of  $\tilde{A}_2$ -groups with respect to the natural generators.

**Theorem 3.2.** [CMS, Th 4.6] *Let  $G$  be an  $\tilde{A}_2$ -group obtained from  $\Pi = PG(2, q)$ . Let  $S$  be the set of natural generators. Then*

$$\kappa(G, S) = \sqrt{2\varepsilon_q},$$

where

$$\varepsilon_q = 1 - \frac{q(\sqrt{q} + \sqrt{q^{-1}} + 1)}{q^2 + q + 1}.$$

The constant  $\sqrt{2\varepsilon_q}$  is equal to the constant in Proposition 3.1. Indeed one can use the proposition to show  $\sqrt{2\varepsilon_q} \leq \kappa(G, S)$  by checking  $S$  satisfies the condition of the proposition, but the other inequality is hard.

We remark that for the above  $G$  and  $S$ , we have

$$\varepsilon(G, S)/|S| = \varepsilon_q,$$

where  $\varepsilon(G, S)$  is the spectral gap. Indeed, we have  $\varepsilon(G, S)/|S| \leq \varepsilon_q$  from Theorem 3.2 and Theorem 2.1. To see the other inequality we first define a graph. Let  $\mathcal{G}(S)$  be the graph whose vertex set is  $S$  such that we join  $s, t \in S$  if there is  $u \in S$  with  $s = tu$ . Assume that  $\mathcal{G}(S)$  is connected. Let  $\lambda$  be the first positive eigenvalue of the Laplacian on  $\mathcal{G}(S)$ . Note that in our case,  $\mathcal{G}(S)$  is the link of a vertex of the building, which is the incidence graph of  $\Pi$ . But it is the incidence graph of the finite projective plane of order  $q$ . So,  $\lambda = 1 - \sqrt{q}/(q + 1)$ , see [BdlHV, Proposition 5.7.6].

Now it is pointed out in [Oz, Example 5] (for the normalized Laplacian), we have (for this we only need that  $\mathcal{G}(S)$  is connected)

$$(2 - \lambda^{-1})|S| \leq \varepsilon(G, S).$$



Indeed, he explicitly gives solutions  $b_i, i \in S$  of the equation 2.2 for  $\varepsilon = (2 - \lambda^{-1})|S|$ , so  $(2 - \lambda^{-1})|S| \leq \varepsilon(G, S)$ . Moreover, the support of  $b_i$  is contained in  $S$ . By computation,  $2 - \lambda^{-1} = \varepsilon_q$ , so we obtain  $\varepsilon_q|S| \leq \varepsilon(G, S)$ . We are done. We have also shown that the equation 2.2 has a solution when  $\varepsilon$  is equal to  $\varepsilon(G, S)$ .

We record this discussion.

**Corollary 3.3.** *Let  $G$  and  $S$  be as in Theorem 3.2, and  $\Delta$  the Laplacian. Then the spectral gap,  $\varepsilon(G, S)$ , (in the sense of Ozawa) is achieved, ie, there is a solution  $b_i, i \in S$  for the equation 2.2 for the  $\varepsilon(G, S)$ . Moreover, the support of  $b_i$  is contained in  $S$ .*

*The set  $S$  satisfies the condition for  $S$  in Proposition 3.1, and the constant in the proposition is actually the Kazhdan constant for  $S$ .*

*Also, for the  $S$  and  $\varepsilon(G, S)$ , the equality holds in the inequality 2.1 in Theorem 2.1.*

Note that Theorem 3.2 is only for the natural generators from triangle presentations. As far as we know, this is the only class of infinite groups whose Kazhdan constants are known.

In view of Corollary 3.3, we are curious to know if the infimum in the definition of  $\kappa(\Gamma, S, \pi)$ ,  $\kappa(\Gamma, S)$  is achieved for the groups in Theorem 3.2.

**3.3. Computation of  $\varepsilon$  of  $\tilde{A}_2$ -groups.** As we mentioned, in the case that  $q = 2, 3$ , there is a list of all  $\tilde{A}_2$ -groups with the natural generators, [CMSZ2].

*The case  $q = 2$ .*

There are 9 presentations:  $A_1, A'_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1$ . From the presentations, we compute lower bounds of the spectral gaps. See Table 1. We know  $\kappa = 0.465175\dots$  by letting  $q = 2$  in Theorem 3.2. We observe that our numerical bound is (almost) identical to the actual value, and the certified bound is also very close. Also, in view of Corollary 3.3, we expect to find solution with  $d = 1$ , ie, the support of  $b_i$  is contained in  $\{1\} \cup S$ , which happens in our experiment.

*The case  $q = 3$ .*

This case contains more groups, and we compute our bounds only for the first several ones: groups of 1.1 to 1.8 in their classification. See Table 2 for the result. We know  $\kappa = 0.687447\dots$  by letting  $q = 3$  in Theorem 3.2. Again, our numerical bound is (almost) identical to the actual value, and the certified bound is also very close.

We see those results as a supporting evidence that if we obtain a lower bound of the spectral gaps, maybe the bound is not so far from the actual value.

For both  $q = 2, 3$ , we found our  $\varepsilon$  for the support range  $d = 1$ . Namely, in the numerical solutions  $b_i$ , the coefficients are non-zero only on the generators. We believe that our solutions,  $\varepsilon$  and  $b_i$ 's, are good numerical approximations of the ones mentioned in [Oz, Example 5]. For example, observe

	certified bound	numerical solution	$\kappa$ by [CMS]
$A_1$	0.465164...	0.465175...	0.465175...
$A'_1$	0.465166...	0.465175...	0.465175...
$A_2$	0.465167...	0.465175...	0.465175...
$A_3$	0.465167...	0.465175...	0.465175...
$A_4$	0.465165...	0.465175...	0.465175...
$B_1$	0.465164...	0.465175...	0.465175...
$B_2$	0.465167...	0.465175...	0.465175...
$B_3$	0.465167...	0.465175...	0.465175...
$C_1$	0.465167...	0.465175...	0.465175...

TABLE 1. Lower bounds of  $\sqrt{2\varepsilon/|S|}$  (therefore  $\kappa$ ) for the groups on the list in [CMSZ2, p. 212] with  $q = 2$ .  $|S| = 14$ . We set  $d = 1$ , then  $m = 7$ .

	certified bound	numerical bound	$\kappa$ by [CMS]
1.1	0.687430...	0.687447...	0.687447...
1.1'	0.687430...	0.687447...	0.687447...
1.2	0.687431...	0.687447...	0.687447...
1.3	0.687430...	0.687447...	0.687447...
1.4	0.687429...	0.687447...	0.687447...
1.5	0.687431...	0.687447...	0.687447...
1.6	0.687433...	0.687447...	0.687447...
1.7	0.687432...	0.687447...	0.687447...
1.8	0.687432...	0.687447...	0.687447...

TABLE 2. Lower bounds of  $\sqrt{2\varepsilon/|S|}$  for some groups on the list in [CMSZ2, pp. 213–222] with  $q = 3$ .  $|S| = 26$ . ( $d = 1$ ,  $m = 13$ .)

$2m = |S|$  holds for our solution, which also holds for the solutions in [Oz, Example 5]

We note that we obtain a positive lower bound for the Kazhdan constants only using the group presentations. In this sense it is a new proof that those groups are Kazhdan groups.

**3.4. Triangles of groups.** We now discuss another family of groups that are lattices on  $\tilde{A}_2$ -buildings. Here is a list:

$$\begin{aligned}
(3.1) \quad G_1 &= \langle a, b, c \mid a^3, b^3, c^3, (ab)^2 = ba, (bc)^2 = cb, (ca)^2 = ac \rangle. \\
G_2 &= \langle a, b, c \mid a^3, b^3, c^3, (ab)^2 = ba, (bc)^2 = cb, (ac)^2 = ca \rangle. \\
G_3 &= \langle a, b, c \mid a^3, b^3, c^3, (ab)^2 = ba, (ac)^2 = ca, (c^{-1}b)^2 = bc^{-1} \rangle. \\
G_4 &= \langle a, b, c \mid a^3, b^3, c^3, (ab)^2 = ba, (ac)^2 = ca, (bc^{-1})^2 = c^{-1}b \rangle.
\end{aligned}$$

	certified bound	numerical bound
$G_1$	0.239014...	0.239146...
$G_2$	0.238700...	0.239146...
$G_3$	0.238405...	0.239146...
$G_4$	0.238605...	0.239146...

TABLE 3. Lower bounds of  $\sqrt{2\varepsilon/|S|}$ , where  $|S| = 6$ , of Ronan's groups (3.1). ( $d = 2$ .) We obtain the same numerical bounds for  $d = 3$ . Theorem 3.2 does not apply to  $S$ , so  $\kappa$  is unknown.

Each of the four groups acts on a  $\tilde{A}_2$ -building such that the action on the triangles is regular and the quotient is one triangle ( $[R]$ ). So, it is a lattice, so that a Kazhdan group. Since the quotient has three vertices, Theorem 3.2 does not apply to this action. Also, the generating set,  $S$  consisting  $a, b, c$  and their inverses, does not satisfy the condition for  $S$  in Proposition 3.1. (For example,  $\mathcal{G}(S)$  is not connected, with three components).

Those four groups are interesting and studied from various viewpoints. Here is a list of facts on the four groups:

- (1) There is a geometric characterization of the four groups, [R, Theorem 2.5]: if  $\Delta$  is a “trivalent triangle geometry” (ie, a 2-dimensional complex of triangles such that the link of each vertex is the incidence graph of  $PG(2, 2)$ ), admitting a group  $G$  of automorphisms that is regular on the set of triangles, then  $G$  is a quotient group of one of  $G_i$ .
- (2) This is the list of all fundamental groups of “triangles of groups” (see [St]) such that the edge groups are  $\mathbb{Z}/3\mathbb{Z}$ , the vertex groups are the Frobenius group of order 21,  $\langle a, b \mid a^3, b^7, aba^{-1} = b^2 \rangle$ , and the face group is trivial.

Each  $G_i$  acts on a  $\tilde{A}_2$ -building properly such that the quotient is one triangle, [Ti, p118,119].

- (3) They are automatic groups (S. M. Gersten and H. Short). They have a common growth function:  $\frac{1+46z+16z^2}{1-8z+16z^2}$ , [FP, Example 5.1].
- (4) They have property (T) since they are lattices of  $\tilde{A}_2$ -buildings.
- (5)  $G_1$  and  $G_3$  are linear, [KMW2].  $G_2$  and  $G_4$  are not arithmetic, [Ti].
- (6)  $G_1$  and  $G_2$  are perfect, [KMW, Prop 1].
- (7)  $G_3$  and  $G_4$  have normal subgroups of index 3, which are  $\Gamma_1$  and  $\Gamma_2$ , resp., as follows:

$$\Gamma_1 = \langle s, t, x \mid s^7 = t^7 = x^7 = 1, st = x, s^3t^3 = x^3 \rangle, \text{ cf. [Es, S 5.3].}$$

$\Gamma_2 = \langle s, t, x \mid s^7 = t^7 = x^7 = 1, st = x^3, s^3t^3 = x \rangle$ , which is not linear, and a subgroup of index 3 in  $G_4$ , [BCL]. So,  $G_4$  is not linear.

This family is interesting for us because of (4). Here are the numerical results of our computation. We obtained positive numbers that are very close to each other:

**Theorem 3.4.** *The spectral gap of  $G_i$  with respect to  $a, b, c$  and their inverses is at least 0.238. More generally, if  $\Delta$  is a trivalent triangle geometry with  $G$  acting as a regular automorphisms on the set of triangles of  $\Delta$ , then the spectral gap of  $G$  w.r.t. the natural three generators and their inverses is at least 0.238.*

*Proof.* The first assertion is clear from our computation. The second assertion follows from the fact (1), since  $G$  is a quotient of one of  $G_i$ , and the spectral gap, does not decrease.  $\square$

Again, our positive bound is obtained only from the presentations, so that it is a new proof that those groups are Kazhdan groups.

Our numerical lower bound 0.239146... is obtained from a numerical lower bound of  $\varepsilon$ , which is 0.171573. We suspect that 0.171573 (and maybe 0.239146... as well) is a good approximation of the spectral gaps (and maybe the Kazhdan constants) for those four groups. In particular, it suggests that those four groups have the same spectral gap (and Kazhdan constants) for the natural generating sets,  $a, b, c$  and their inverses.

It seems there is no lower bound explicitly given for the Kazhdan constants of those four groups w.r.t.  $S = \{a^\pm, b^\pm, c^\pm\}$ . As we said  $S$  does not satisfy the condition in Proposition 3.1. It is possible to obtain a generating set  $S'$  using the proposition, for which we have  $\kappa(G_i, S') = 0.465175...$ . That would give a lower bound of  $\kappa(G_i, S)$ , by writing each element of  $S'$  as a product of elements of  $S$ , but it will be much smaller than 0.465175..., in particular smaller than our lower bound.

We point out that our numerical bound for  $\varepsilon$  is almost identical to  $(\sqrt{2} - 1)^2 = 0.17157287525...$ , and that the numerical bound for  $\kappa$  is almost identical to  $(\sqrt{2} - 1)/\sqrt{3} = 0.23914631173...$ . We do not know any explanation. Since our numerical bound is in fact a “solution”, so we believe

$$\varepsilon = (\sqrt{2} - 1)^2$$

holds.

#### 4. $\mathrm{SL}(3, \mathbb{Z})$ , $\mathrm{SL}(4, \mathbb{Z})$ AND $\mathrm{St}_3(\mathbb{Z})$

Next we deal with  $\mathrm{SL}(n, \mathbb{Z})$ . [NT] obtained a spectral gap for  $\mathrm{SL}(3, \mathbb{Z})$ . Let  $E_n$  be the set of all  $n \times n$  elementary matrices with  $\pm 1$  off the diagonal. This is a symmetric generating set of  $\mathrm{SL}(n, \mathbb{Z})$ .

An upper bound of  $\kappa(\mathrm{SL}(n, \mathbb{Z}), E_n)$  was given by Zuk (see [Sha, p. 149]), and a lower bound by Shalom [Sha]. Kassabov improved the lower bound in [Kas1].

**Theorem 4.1** (Kassabov [Kas1], Zuk). *For  $\mathrm{SL}(n, \mathbb{Z})$  and the symmetric generating set  $E_n$ , the (optimal) Kazhdan constant  $\kappa(\mathrm{SL}(n, \mathbb{Z}), E_n)$  satisfies*

$$\frac{1}{42\sqrt{n} + 860} \leq \kappa(\mathrm{SL}(n, \mathbb{Z}), E_n) \leq \sqrt{\frac{2}{n}}.$$

For  $\mathrm{SL}(3, \mathbb{Z})$ , Netzer and Thom [NT] already obtained a lower bound that is much better than the previous ones by solving a semidefinite programming based on Theorem 2.2. We improve their bound. For  $\mathrm{SL}(4, \mathbb{Z})$  we obtain a new bound that is much better than any other known bounds. Our algorithm could not find any bound for  $\mathrm{SL}(5, \mathbb{Z})$ . The results are summarized in Table 4.

	Kassabov	Netzer-Thom	Certified bound	Upper bound by Zuk
$\mathrm{SL}(3, \mathbb{Z})$	0.001072...	0.1783...	0.2155...	0.8164...
$\mathrm{SL}(4, \mathbb{Z})$	0.001059...		0.3285...	0.7071...

TABLE 4. Lower bounds of  $\kappa(\mathrm{SL}(n, \mathbb{Z}), E_n)$  for  $n = 3, 4$ . ( $d = 2$ )

Here we explain some details. As in [NT], we try to find solutions  $\varepsilon$  and  $b_i$ 's of the equation (2.2) by numerical calculation for the support range  $d = 2$  (with respect to  $E_n$ ). There are 121 group elements with the word length  $\leq 2$  in  $\mathrm{SL}(3, \mathbb{Z})$  and 433 in  $\mathrm{SL}(4, \mathbb{Z})$ . For  $d = 2$ , the certified bound  $\varepsilon$  is 0.278648... for  $\mathrm{SL}(3, \mathbb{Z})$  (0.1905 in [NT]). Our numerical bound is 0.2804..., and 1.29562... for  $\mathrm{SL}(4, \mathbb{Z})$  (the numerical bound is 1.313...). From the certified bounds for  $\varepsilon$ , we obtain our bounds for the Kazhdan constant.

We discuss the Steinberg groups,  $\mathrm{St}_n(\mathbb{Z})$ , which are defined as follows:

$$\langle x_{ij} (i, j \in \{1, 2, \dots, n\}, i \neq j) \mid [x_{ij}, x_{jk}] = x_{ik} (i \neq k), [x_{ij}, x_{kl}] = 1 (i \neq l, j \neq k) \rangle.$$

It is known that for  $n \geq 3$ ,  $\mathrm{St}_n(\mathbb{Z})$  is an extension of  $\mathrm{SL}(n, \mathbb{Z})$  by  $\mathbb{Z}/2\mathbb{Z}$ , (see [Mil, Th 10.1]), therefore  $\mathrm{St}_n(\mathbb{Z})$  is a Kazhdan group for  $n \geq 3$ . In fact,  $\mathrm{SL}(n, \mathbb{Z})$  is obtained from  $\mathrm{St}_n(\mathbb{Z})$  by adding one relation:  $(x_{12}x_{21}^{-1}x_{12})^4 = 1$ . This element has order 2. The generators  $x_{ij}$  are mapped to the natural generators,  $E_n$ , of  $\mathrm{SL}(n, \mathbb{Z})$ . So,  $\kappa(\mathrm{St}_n(\mathbb{Z}), \{x_{ij}\}) \leq \kappa(\mathrm{SL}(n, \mathbb{Z}), E_n)$ .

By our computation, we obtain a certified lower bound as follows:

$$(4.1) \quad 0.171028... \leq \kappa(\mathrm{St}_3(\mathbb{Z}), \{x_{ij}\}).$$

We only use the presentation of the group, so this gives a new proof that  $\mathrm{St}_3(\mathbb{Z})$  is a Kazhdan group. But the computation did not finish for  $\mathrm{St}_4(\mathbb{Z})$ .

## 5. FINITE REFLECTION GROUPS

Now we turn our attention to finite groups. All finite groups are Kazhdan groups, (cf. [BdlHV]), but to find the Kazhdan constants or the spectral gaps are not easy at all and they are known only for certain families, for example, finite cyclic groups [BH], finite Coxeter groups [Kas2], with respect to natural generating sets. In this section, we compare these results and computer calculations. We also examine some finite groups whose Kazhdan constants are unknown.

**5.1. Coxeter groups.** It is a classical fact that finite irreducible Coxeter groups are classified by Dynkin diagrams. For Dynkin diagrams  $A_n, B_n, \dots$ , we denote the corresponding Coxeter groups by the same symbols by abuse of notation. We denote their Coxeter generators as  $S_{A_n}, S_{B_n}, \dots$ .

The Kazhdan constants are known.

**Theorem 5.1** (Kassabov [Kas2, Remark 6.3], Theorem and Proposition 4 of [BH], Theorems 2.1, 2.2 of [Bag] for  $A_n, B_n$  and  $I_2(n)$ ).

$$\begin{aligned}\kappa(A_n, S_{A_n}) &= \sqrt{\frac{24}{(n+1)^3 - (n+1)}}, \\ \kappa(B_n, S_{B_n}) &= \sqrt{\frac{12}{n(4 - 3\sqrt{2} + 3(\sqrt{2} - 1)n + 2n^2)}}, \\ \kappa(D_n, S_{D_n}) &= \sqrt{\frac{12}{n(n-1)(2n-1)}}, \\ \kappa(F_4, S_{F_4}) &= \sqrt{\frac{14 - 9\sqrt{2}}{34}}, \quad \kappa(E_6, S_{E_6}) = \sqrt{\frac{1}{39}}, \quad \kappa(E_7, S_{E_7}) = \sqrt{\frac{4}{399}}, \\ \kappa(E_8, S_{E_8}) &= \sqrt{\frac{1}{310}}, \quad \kappa(H_3, S_{H_3}) = \sqrt{\frac{124 - 48\sqrt{5}}{241}}, \quad \kappa(H_4, S_{H_4}) = \sqrt{\frac{83 - 36\sqrt{5}}{409}}, \\ \kappa(I_2(n), S_{I_2(n)}) &= 2 \sin\left(\frac{\pi}{2n}\right) \quad (n \geq 3, \text{ dihedral group of order } 2n)\end{aligned}$$

Kassabov also computed the spectral gaps of finite Coxeter groups.

**Theorem 5.2** (Kassabov [Kas2, Remark 6.3]). *The spectral gap of the unnormalized Laplacian  $\Delta$  is*

$$\varepsilon = 4 \left(1 - \cos \frac{\pi}{h}\right)$$

where  $h$  is the Coxeter number.

Here  $h = n+1, 2n, 2(n-1), 12, 18, 30, 12, 10, 30, m$  for  $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$  respectively.

Notice that the left hand side of the inequality (2.1) is  $\sqrt{\frac{4}{n}(1 - \cos \frac{\pi}{h})}$ , which is not equal to the Kazhdan constant.

We compute lower bounds of the spectral gaps, then in our usual manner give lower bounds of the Kazhdan constants. See Tables 5, 6, 7, 8, for the results. Our numerical bounds for  $\sqrt{2\varepsilon/|S|}$  are identical to the actual value (ie. our numerical bound for  $\varepsilon$  is identical to the actual constant by Kassabov).

**5.2. Complex reflection groups.** Next we check (finite) complex reflection groups. The irreducible (ie, not a product) ones are classified into an infinite families  $G(m, p, n)$  and 34 exceptional cases. We apply our algorithms to some of them.

	certified bound	numerical bound	$\sqrt{2\varepsilon/ S }$	$\kappa$
$A_2$	0.99985...	1.00000...	1	1.00000...
$A_3$	0.62341...	0.62491...	0.62491...	0.63245...
$A_4$	0.43661...	0.43701...	0.43701...	0.44721...
$A_5$	0.32625...	0.32738...	0.32738...	0.33806...
$A_6$	0.25601...	0.25694...	0.25694...	0.26726...
$A_7$	0.20818...	0.20856...	0.20856...	0.21821...
$A_8$	0.17334...	0.17364...	0.17364...	0.18257...
$\vdots$				

TABLE 5. Coxeter groups  $A_n$ . Lower bounds of  $\sqrt{2\varepsilon/|S|}$ , and the known values by Kassabov. ( $d = 2$ )

	certified	expected	$\sqrt{2\varepsilon/ S }$	$\kappa$
$B_2$	0.76482...	0.76536...	0.76536...	0.76536...
$B_3$	0.42163...	0.42264...	0.42264...	0.43147...
$B_4$	0.27464...	0.27589...	0.27589...	0.28580...
$B_5$	0.19718...	0.19787...	0.19787...	0.20707...
$B_6$	0.14872...	0.15071...	0.15071...	0.15889...
$B_7$	0.11558...	0.11969...	0.11969...	0.12689...
$B_8$	0.09053...	0.09801...	0.09801...	0.10437...
$\vdots$				

TABLE 6. Coxeter groups  $B_n$ . ( $d = 3$ )

	certified bound	numerical bound	$\sqrt{2\varepsilon/ S }$	$\kappa$
$D_4$	0.36556...	0.36602...	0.36602...	0.37796...
$D_5$	0.24553...	0.24677...	0.24677...	0.25819...
$D_6$	0.18044...	0.18063...	0.18063...	0.19069...
$D_7$	0.13805...	0.13953...	0.13953...	0.14824...
$D_8$	0.11146...	0.11196...	0.11196...	0.11952...
$\vdots$				

TABLE 7. Coxeter groups  $D_n$ . ( $d = 2$ )

Let  $\mathfrak{S}_n$  be the symmetric group of  $n$  elements. For  $\sigma \in \mathfrak{S}_n$  and  $a_1, \dots, a_n \in \mathbb{C}$ , we let  $[(a_1, \dots, a_n), \sigma]$  be the  $n \times n$  matrix whose  $(i, j)$ -entry is  $a_i$  if  $(i, j) = (i, \sigma(i))$  and 0 otherwise. For  $m, p, n \in \mathbb{N}$  with  $p|m$ , let

$$G(m, p, n) = \{ [(a_1, \dots, a_n), \sigma] \mid \sigma \in \mathfrak{S}_n, a_i \in \mathbb{C}, a_i^m = 1, \left( \prod_{j=1}^n a_j \right)^{m/p} = 1 \},$$

which is a finite subgroup of  $U(n)$ .

	certified bound	numerical bound	$\sqrt{2\varepsilon/ S }$	$\kappa$
$E_6$	0.15032...	0.15071...	0.15071...	0.16012...
$E_7$	0.09203...	0.09317...	0.09317...	0.10012...
$E_8$	0.05164...	0.05233...	0.05233...	0.05679...
$F_4$	0.18334...	0.18459...	0.18459...	0.19342...
$H_3$	0.25520...	0.25545...	0.25545...	0.26299...
$H_4$	0.07316...	0.07401...	0.07401...	0.07820...

TABLE 8. Coxeter groups of exceptional types.  $d = 2$  for  $E_n$  (simply laced) and  $d = 3$  for the others.

By definition,  $G(m, 1, n)$  is isomorphic to the wreath product  $(\mathbb{Z}/m\mathbb{Z})^n \wr \mathfrak{S}_n$ , and  $G(m, p, n)$  is an index  $p$  subgroup of it. Let  $\zeta_m = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$ .

The following set generates  $G(m, p, n)$ .

- $G(m, 1, n)$ :

$$[(\zeta_m, 1, \dots, 1), \text{id}], \quad [(1, \dots, 1), (i, i+1)] \quad (i = 1, \dots, n-1)$$

- $G(m, m, n)$ :

$$[(\zeta_m^{-1}, \zeta_m, 1, \dots, 1), (1, 2)], \quad [(1, \dots, 1), (i, i+1)] \quad (i = 1, \dots, n-1)$$

- $G(m, p, n)$  ( $1 < p < m$ ,  $p|m$ ):

$$[(\zeta_{m/p}, 1, \dots, 1), \text{id}], \quad [(\zeta_m^{-1}, \zeta_m, 1, \dots, 1), (1, 2)], \\ [(1, \dots, 1), (i, i+1)] \quad (i = 1, \dots, n-1)$$

Moreover, a group presentation with respect to this generating system has been obtained in [BMR, Proposition 3.2]. In this paper, we denote this generating system by  $S_{G(m,p,n)}$ . As groups with generators, we have

$$(G(1, 1, n), S_{G(1,1,n)} \setminus \{1\}) \cong (A_{n-1}, S_{A_{n-1}}), \quad (G(2, 1, n), S_{G(2,1,n)}) \cong (B_n, S_{B_n}), \\ (G(2, 2, n), S_{G(2,2,n)}) \cong (D_n, S_{D_n}), \quad (G(m, m, 2), S_{G(m,m,2)}) \cong (I_2(m), S_{I_2(m)}).$$

It seems the Kazhdan constant is unknown for complex reflection groups, but the Kazhdan constants,  $\hat{\kappa}$ , for  $G(m, 1, n)$  for irreducible representations (namely, in the definition we only look at irreducible representations) are known. Define

$$\hat{\kappa}(\Gamma, S) = \inf\{\kappa(\Gamma, S, \pi) \mid \pi \text{ is irreducible}\}.$$

From the definition, we have  $\kappa \leq \hat{\kappa}$ .

**Theorem 5.3** ([Bag]).

$$\hat{\kappa}(G(m, 1, n), S_{G(m,1,n)}) = \sqrt{\frac{|1 - \zeta_m|^2}{\sum_{j=1}^n \left(1 + \frac{|1 - \zeta_m|}{\sqrt{2}}(j-1)\right)^2}}$$



Now, here is the result of our computation for  $G(m, 1, n)$ , Table 9. For  $G(m, 1, n)$  there is a gap between our certified/numerical bound and  $\hat{\kappa}$ , so that we suspect the equality does not hold in inequality 2.1, at least for those families.

	certified bound	numerical bound	$\hat{\kappa}$ by Bagno
$G(3, 1, 2)$	0.68040...	0.68177...	0.71010...
$G(3, 1, 3)$	0.38644...	0.38753...	0.40997...
$G(3, 1, 4)$	0.25605...	0.25749...	0.27490...
$G(3, 1, 5)$	0.18544...	0.18685...	0.20067...
$G(3, 1, 6)$	0.14080...	0.14352...	0.15476...
$G(3, 1, 7)$	0.10850...	0.11469...	0.12405...
$\vdots$			
$G(4, 1, 2)$	0.62387...	0.62491...	0.63245...
$G(4, 1, 3)$	0.36392...	0.36602...	0.37796...
$G(4, 1, 4)$	0.24449...	0.24677...	0.25819...
$G(4, 1, 5)$	0.17895...	0.18063...	0.19069...
$\vdots$			
$G(5, 1, 2)$	0.55741...	0.56301...	0.56341...
$G(5, 1, 3)$	0.34022...	0.34078...	0.34752...
$G(5, 1, 4)$	0.23243...	0.23374...	0.24173...
$\vdots$			
$G(6, 1, 2)$	0.50135...	0.50462...	0.50544...
$G(6, 1, 3)$	0.31341...	0.31469...	0.32037...
$G(6, 1, 4)$	0.21875...	0.21964...	0.22654...
$\vdots$			

TABLE 9. Lower bounds for  $\sqrt{2\varepsilon/|S|}$  and  $\hat{\kappa}$  for  $G(m, 1, n)$ . ( $d = 3$ )

Next, we check  $G(m, m, n)$ . To compare with our numerical results, we give an upper bound of  $\kappa(G(m, m, n), S_{G(m, m, n)})$ .

**Proposition 5.4** (Upper bound of  $\kappa$ ). *For  $m \geq 2$  and  $n \geq 2$ , we have*

$$\kappa(G(m, m, n), S_{G(m, m, n)}) \leq \sqrt{\frac{2|1 - \zeta_{2m}|^2}{2 + \sum_{j=1}^{n-2} |1 + |1 - \zeta_{2m}|j|^2}}$$

We remark that when  $n = 2$ , although  $(G(m, m, 2), S_{G(m, m, 2)}) \cong (D_n, S_{D_n})$ , the bound does not coincide with  $\kappa(D_n, S_{D_n})$ .

*Proof.* If we let

$$\eta = (1, \zeta_{2m}, \zeta_{2m} + \zeta_{2m}|1 - \zeta_{2m}|, \dots, \zeta_{2m} + \zeta_{2m}|1 - \zeta_{2m}|(n-2)),$$

then  $\|\eta\|^2 = 2 + \sum_{j=1}^{n-2} |1 + |1 - \zeta_{2m}|j|^2$  and  $\|s \cdot \eta - \eta\|^2 = 2|1 - \zeta_{2m}|^2$  for all  $s \in S_{G(m, m, n)}$ . So we obtain the desired upper bound.  $\square$

Our numerical results for  $G(m, m, n)$  are in Table 10

	certified bound	numerical bound	upper bound in Prop 5.4
$G(3, 3, 2)$	0.99999...	1.00000...	1.00000...
$G(3, 3, 3)$	0.55621...	0.55851...	0.70710...
$G(3, 3, 4)$	0.34256...	0.34350...	0.53452...
$G(3, 3, 5)$	0.23364...	0.23596...	0.42640...
$G(3, 3, 6)$	0.16757...	0.17445...	0.35355...
$\vdots$			
$G(4, 4, 2)$	0.76482...	0.76536...	0.76536...
$G(4, 4, 3)$	0.46141...	0.46538...	0.55780...
$G(4, 4, 4)$	0.30330...	0.30528...	0.43136...
$\vdots$			
$G(5, 5, 2)$	0.61594...	0.61803...	0.61803...
$G(5, 5, 3)$	0.38720...	0.38968...	0.45950...
$G(5, 5, 4)$	0.26542...	0.26714...	0.36124...
$\vdots$			

TABLE 10. Lower bounds of  $\sqrt{2\varepsilon/|S|}$  for  $G(3, 3, n)$ ;  $G(4, 4, n)$ ; and  $G(5, 5, n)$ . ( $d = 3$  for  $n = 2$ ,  $d = 4$  for  $n = 3, 4$ ).

We check a few more families for  $G(m, p, n)$ , which is in Table 11.

	certified bound	numerical bound
$G(4, 2, 2)$	0.91909...	0.91940...
$G(4, 2, 3)$	0.49128...	0.49288...
$G(4, 2, 4)$	0.30769...	0.30883...
$G(4, 2, 5)$	0.21409...	0.21585...
$\vdots$		
$G(6, 2, 2)$	0.77478...	0.77740...
$G(6, 2, 3)$	0.42686...	0.43170...
$G(6, 2, 4)$	0.27576...	0.27775...
$\vdots$		
$G(6, 3, 2)$	0.81608...	0.81649...
$G(6, 3, 3)$	0.45887...	0.46055...
$G(6, 3, 4)$	0.29070...	0.29538...
$\vdots$		

TABLE 11. Lower bounds of  $\sqrt{2\varepsilon/|S|}$  for  $G(4, 2, n)$ . ( $d = 3$ );  $G(6, 2, n)$  ( $d = 3$ );  $G(6, 3, n)$  ( $d = 3$  for  $n = 2$ ,  $d = 4$  for  $n = 3, 4$ ).

6.  $\mathrm{SL}(n, \mathbb{F}_p)$ 

Our last examples are  $\mathrm{SL}(n, \mathbb{F}_p)$ . The Kazhdan constants are not known. But the following is known.

**Theorem 6.1** (Kassabov [Kas1, Theorem A’]). *The Kazhdan constant for  $\mathrm{SL}(n, \mathbb{F}_p)$  with respect to the set  $E_n$  of elementary matrices with  $\pm 1$  off the diagonal satisfies*

$$(6.1) \quad \kappa(\mathrm{SL}(n, \mathbb{F}_p), E_n) \geq \frac{1}{31\sqrt{n} + 700}.$$

We compute lower bounds for  $\mathrm{SL}(n, \mathbb{F}_p)$  with respect to the set  $E_n$  with  $n, p$  small. Using this example, we give an idea on how our bounds possibly depend on the support range  $d$ . Table 12 is for the support range  $d = 2$ , and Table 13 is for the support range  $d = 3$ . We may obtain a better (ie, larger) bound for a larger  $d$ , but there is more chance that the computation does not finish. For example, the bound for  $\mathrm{SL}(2, \mathbb{F}_5)$  improves much if we change  $d = 2$  to  $d = 3$ .

$p$	3	5	7	$\kappa_0$
$\mathrm{SL}(2, \mathbb{F}_p)$	0.7961...	0.2580...	(*)	0.001344...
$\mathrm{SL}(3, \mathbb{F}_p)$	0.6716...	0.4981...	0.3508...	0.001326...
$\mathrm{SL}(4, \mathbb{F}_p)$	0.5974...	0.4812...	0.3284...	0.001312...

TABLE 12. Certified lower bounds of the Kazhdan constant ( $d = 2$ ), and Kassabov’s lower bound  $\kappa_0$ , the right hand side of the inequality (6.1). Our numerical bound for  $\mathrm{SL}(2, \mathbb{F}_7)$  is close to 0, so that we could not get a positive number as a certified bound (\*).

$p$	3	5	7	$\kappa_0$
$\mathrm{SL}(2, \mathbb{F}_p)$	0.7958...	0.6145...	0.5387...	0.001344...
$\mathrm{SL}(3, \mathbb{F}_p)$	0.6683...	0.4410...	(*)	0.001326...

TABLE 13. Certified lower bounds of the Kazhdan constant. ( $d = 3$ .) The computation did not finish for  $\mathrm{SL}(3, \mathbb{F}_7)$  (\*).

## 7. PROBLEMS SUGGESTED BY THE EXPERIMENTAL RESULTS

We mention problems that naturally arise from our experiments.

- (1) Is our bound, 0.239, for the Kazhdan constants of  $G_1, G_2, G_3, G_4$  in Section 3.3 close to the actual values? Do the four groups have same Kazhdan constants? Is the Kazhdan constant equal to  $(\sqrt{2} - 1)/\sqrt{3} = 0.23914631173...$  ?

Similarly, is our bound 0.171573 close to the spectral gaps  $\varepsilon$  of the four groups? Do they have the same spectral gaps? Is it  $(\sqrt{2}-1)^2 = 0.17157287525\dots$ ?

To answer the first question for  $G_i$ , we only need to find one representation  $\pi$  and one vector  $\xi \in \mathcal{H}$  such that  $\max_{s \in S} \|\pi(s)\xi - \xi\|$  is close to 0.239, where  $S$  consists of  $a, b, c$  and the inverses. If one finds such representation, it also implies that 0.171573 is close to  $\varepsilon$ .

The numerical solutions  $b_i$  we find for the above spectral gap are for  $d = 2$ , ie, the support of  $b_i$  are contained in  $\{1\} \cup S \cup S^2$ . Is there any explanation for that? (cf. Corollary 3.3, where we find a solution for  $d = 1$ .)

- (2) Among  $\mathrm{SL}(n, \mathbb{Z})$ ,  $n \geq 3$ , is the spectral gap  $\varepsilon$  a monotone decreasing function on  $n$  w.r.t. the generating sets of elementary matrices? How about the Kazhdan constants? Our lower bound for  $\mathrm{SL}(4, \mathbb{Z})$  is larger than the one for  $\mathrm{SL}(3, \mathbb{Z})$ . If it is not monotone, for which  $n$ , are the Kazhdan constant of  $\mathrm{SL}(n, \mathbb{Z})$  largest? Combining our lower bound for  $\mathrm{SL}(4, \mathbb{Z})$  and Zuk's upper bound, it must be at most 18.
- (3) Does the equality hold in the inequality 2.1 for finite complex reflection groups? In view of this problem, compute the spectral gaps (maybe using the method by Kassabov, see [Kas2, Remark 6.5]).

We believe that our (in particular, numerical) bounds for the spectral gap are very close to the actual values, and that our results suggest that the equality does not hold.

- (4) What is the behavior of the spectral gap (or the Kazhdan constant) of  $\mathrm{SL}(n, \mathbb{F}_q)$  when  $q$  increases with  $n$  fixed? Does it converge to the spectral gap (or the Kazhdan constant) of  $\mathrm{SL}(n, \mathbb{Z})$  as  $q \rightarrow \infty$ ? Notice that the answer is negative for the Kazhdan constant for  $n = 2$ . This is because  $\mathrm{SL}(2, \mathbb{Z})$  does not have property (T), so that  $\kappa = 0$ , while there is a uniform positive lower bound for  $\mathrm{SL}(2, \mathbb{F}_p, E_2)$  by Kassabov (see the discussion around the end of the introduction in [BdlHV]).

How about when  $n$  increases with  $q$  fixed? Our experiments may suggest that they are monotone on  $q$  with  $n$  fixed; and also on  $n$  with  $q$  fixed.

- (5) For a given  $\varepsilon < \varepsilon(G, S)$ , the equation 2.2 has solutions  $b_i$  ([Oz]). But is there a solution for  $\varepsilon(G, S)$ ? See Corollary 3.3.

Is it possible to estimate the support range  $d$  of the solutions in advance? Is there a universal upper bound of  $d$  for all  $\varepsilon$ ?

## REFERENCES

- [BH] Roland Bacher and Pierre de la Harpe, *Exact values of Kazhdan constants for some finite groups*, J. Algebra 163 (1994) 495–515.
- [BCL] Uri Bader, Pierre-Emmanuel Caprace, Jean Lecureux, *On the linearity of lattices in affine buildings and ergodicity of the singular Cartan flow*, arXiv:1608.06265

- [Bag] Eli Bagno, *Kazhdan constants of some colored permutation groups*, Journal of Algebra 282 (2004) 205–231.
- [BS] W. Ballmann, J. Świątkowski, *On  $L^2$ -cohomology and property (T) for automorphism groups of polyhedral cell complexes*, Geom. Funct. Anal. 7 (1997), no. 4, 615–645.
- [BdlHV] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, 11. Cambridge University Press, Cambridge, 2008. xiv+472 pp.
- [BMR] Michel Broué, Gunter Malle, and Raphaël Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. 500 (1998), 127–190.
- [BZ] Nathaniel P. Brown and Narutaka Ozawa,  *$C^*$ -algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.
- [CMSZ] D I Cartwright, A M Mantero, T Steger, A Zappa, *Groups acting simply transitively on the vertices of a building of type  $\tilde{A}_2$ , I*, Geom. Dedicata 47 (1993), no. 2, 143–166.
- [CMSZ2] D I Cartwright, A M Mantero, T Steger, A Zappa, *Groups acting simply transitively on the vertices of a building of type  $\tilde{A}_2$ , II: The cases  $q = 2$  and  $q = 3$* , Geom. Dedicata 47 (1993), no. 2, 167–223.
- [CMS] D I Cartwright, W Młotkowski, T Steger, *Property (T) and  $\tilde{A}_2$  groups*, Ann. Inst. Fourier (Grenoble) 44 (1994) 213–248.
- [Es] Jan Essert, *A geometric construction of panel-regular lattices for buildings of types  $\tilde{A}_2$  and  $\tilde{C}_2$* , Algebr. Geom. Topol. 13 (2013), no. 3, 1531–1578.
- [FP] William Floyd, Walter Parry, *The growth of nonpositively curved triangles of groups*, Invent. Math. 129 (1997), no. 2, 289–359.
- [Kas1] Martin Kassabov, *Kazhdan constants for  $SL_n(\mathbb{Z})$* , Internat. J. Algebra Comput. 15 (2005), no. 5–6, 971–995.
- [Kas2] M. Kassabov, *Subspace arrangements and property T*, Groups Geom. Dyn. 5 (2011), no. 2, 445–477.
- [Ka] Ryota Kawakami. *Kazhdan’s Property(T) and Semi-Definite Programming*. Master Thesis paper, 2015, Kyoto University.
- [KMW] Peter Köhler, Thomas Meixner, Michael Wester, *Triangle groups*, Comm. Algebra 12 (1984), no. 13–14, 1595–1625.
- [KMW2] Peter Köhler, Thomas Meixner, Michael Wester, *The affine building of type  $\tilde{A}_2$  over a local field of characteristic two*, Arch. Math. (Basel) 42 (1984), no. 5, 400–407.
- [Oz] Narutaka Ozawa, *Noncommutative real algebraic geometry of Kazhdan’s property (T)*, arXiv:1312.5431
- [Mil] John Milnor, *Introduction to algebraic K-theory*. Annals of Mathematics Studies, No. 72. Princeton University Press, xiii+184 pp.
- [NT] Tim Netzer and Andreas Thom, *Kazhdan’s Property (T) via Semidefinite Optimization*, arXiv:1411.2488
- [Pan] Pierre Pansu, *Formules de Matsushima, de Garland et propriété (T) pour des groupes agissant sur des espaces symétriques ou des immeubles*, Bull. Soc. Math. France 126 (1998), no. 1, 107–139.
- [R] M. A. Ronan, *Triangle geometries*, J. Combin. Theory Ser. A 37 (1984), no. 3, 294–319.
- [Sha] Yehuda Shalom, *Bounded generation and Kazhdan’s property (T)*, Publications Mathématiques de l’IHÉS 90 (1999), 145–168.
- [ST] G. Shephard and J. Todd, *Finite unitary reflection groups*, Canadian J. Math. 6, (1954). 274–304.
- [Sp] David Speyer, *How feasible is it to prove Kazhdan’s property (T) by a computer?*, an answer to MathOverflow question, <http://mathoverflow.net/a/154459>

- [St] John R. Stallings, *Non-positively curved triangles of groups*, Group theory from a geometrical viewpoint (Trieste, 1990), 491–503, World Sci. Publ., River Edge, NJ, 1991.
- [Ti] Jacques Tits, *Buildings and group amalgamations*, Proceedings of groups at St. Andrews 1985, 110–127, London Math. Soc. Lecture Note Ser., 121, Cambridge Univ. Press, Cambridge, 1986.
- [Zuk] A. Zuk, *La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres*, C. R. Acad. Sci. Paris 323, Serie I (1996), 453–458.